Towards Combinatory Logic Synthesis

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Based on joint work with
B. Düdder, M. Martens (Dortmund), P. Urzyczyn (Warsaw)
and with thanks to Roger Hindley

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This Talk

- The applied research programme behind recent theoretical works on inhabitation [1] [2] [3]
- Behavioural type specification with intersection types and combinatory logic for composition synthesis

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[1] B.Düdder, M.Martens, J.Rehof, P. Urzyczyn: *Bounded Combinatory Logic*. CSL Computer Science Logic, 2012.

[2] J.Rehof, P.Urzyczyn: *Finite Combinatory Logic with Intersection Types*. TLCA Typed Lambda Calculi and Applications, 2011.

[3] J. Rehof, P. Urzyczyn: *The Complexity of Inhabitation with Explicit Intersection*. Logic and Program Semantics. Essays Dedicated to Dexter Kozen, 2012.

Combinatory Logic Synthesis

The typability relation in combinatory logic (CL)

$$\Gamma \vdash e : \tau$$

as logical model of applicative composition e of named *component interfaces* ($\mathtt{F} : \sigma$) from a *repository* Γ satisfying a specification τ.

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$$\frac{}{\Gamma, X : A \vdash_s X : S(A)}(\mathsf{var}) \qquad \frac{\Gamma \vdash_s e : A \to B \quad \Gamma \vdash_s e' : A}{\Gamma \vdash_s (e\ e') : B}(\to\mathsf{E})$$

Example Repository

$$
\begin{array}{lcl}
\texttt{Tr} & : & () \to \mathtt{D}((\mathtt{R},\mathtt{R}),\mathtt{R},\mathtt{R}) \\
\texttt{pos} & : & \mathtt{D}((\mathtt{R},\mathtt{R}),\mathtt{R},\mathtt{R}) \to ((\mathtt{R},\mathtt{R}),\mathtt{R}) \\
\texttt{cdn} & : & ((\mathtt{R},\mathtt{R}),\mathtt{R}) \to (\mathtt{R},\mathtt{R}) \\
\texttt{fst} & : & (\mathtt{R},\mathtt{R}) \to \mathtt{R} \\
\texttt{snd} & : & (\mathtt{R},\mathtt{R}) \to \mathtt{R} \\
\texttt{tmp} & : & \mathtt{D}((\mathtt{R},\mathtt{R}),\mathtt{R},\mathtt{R}) \to \mathtt{R} \\
\texttt{cc2pl} & : & (\mathtt{R},\mathtt{R}) \to (\mathtt{R},\mathtt{R}) \\
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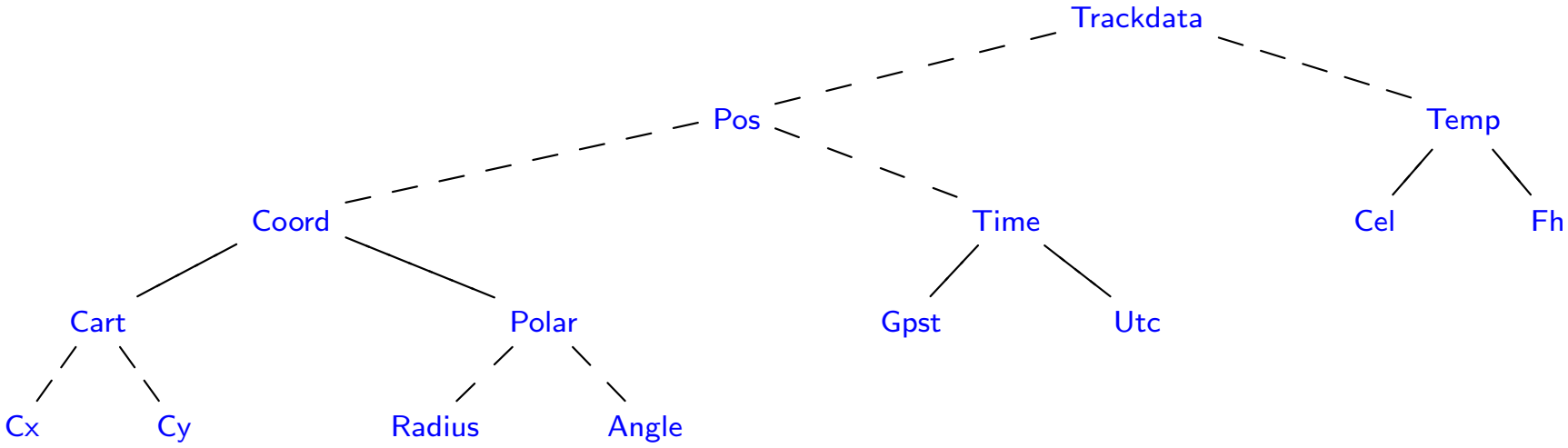
\end{array}
$$

Semantics: concepts and behaviour



Semantic repository: superposition with $\cap$

$$
\begin{array}{lcl}
\Gamma = \{ & & \\
\texttt{Tr} & : & () \to \texttt{D}((\texttt{R},\texttt{R}) \cap \mathit{Cart}, \texttt{R} \cap \mathit{Gpst}, \texttt{R} \cap \mathit{Cel}) \\
\texttt{pos} & : & \texttt{D}((\texttt{R},\texttt{R}) \cap \varepsilon, \texttt{R} \cap \eta, \texttt{R}) \to ((\texttt{R},\texttt{R}) \cap \varepsilon, \texttt{R} \cap \eta) \cap \mathit{Pos} \\
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Synthesis via inhabitation

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Overview

- Theoretical background
- Composition synthesis
- Examples

Inhabitation in Combinatory Logic

- Usually, in CL, one considers fixed *base* of combinators understood as axiom *schemes*
- Example: **SK**-calculus
  - $\mathbf{K} : \alpha \to \beta \to \alpha$
  - $\mathbf{S} : (\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to (\alpha \to \gamma)$
- Logically, such systems are Hilbert-style proof systems: modus ponens + schematism
- Under Curry-Howard isomorphism, axiom schemes are polymorphic types, and provability is the question of *inhabitation*: $\Gamma \vdash ? : \tau$
- Example: Inhabitation in **SK**-calculus ($\sim$ simple typed λ-calculus) is $\textsc{Pspace}$ complete (Statman 1979).

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- **Relativized inhabitation**: the set of combinators is *not fixed* but given as part of input
  - *Given* Γ *and* τ*, does there exist applicative term* e *such that* $\Gamma \vdash e : \tau$*?*

Synthesis: Relativized Inhabitation

- Since repositories Γ vary and each may change, fixed-base problem is not the right model for synthesis
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- The relativized inhabitation problem for CL is *undecidable*, even in simple types (Linial-Post theorem)

Linial-Post Theorems

- Tarski (Princeton Bicentennial 1946): decision problem (provability) for arbitrary propositional calculi (PPC)?
- L. Linial and E. Post, 1948: there exists an undecidable PPC [*Bulletin of the AMS* (55) 1949]
- Gladstone [1965], Singletary [1967, 1974]: implicational fragment of PPC can represent every r.e. many-one degree
- Later, many theories have been considered in combinatory logic (**B**, **B**′ and friends). Some turned out to be surprisingly complicated (e.g. relevance logic, ticket entailment)

Two-counter Automata

$\mathcal{A} = \langle Q, q_0, q_F, \delta \rangle$

- Control states Q (start state q_0, accepting state q_F)
- Counters $\mathtt{c}_1$ and $\mathtt{c}_2$
- Transition relation δ given by the instructions ($i = 1, 2$)
  - $\mathtt{q} : \mathtt{c}_i := \mathtt{c}_i + 1$; `goto` p
  - $\mathtt{q} : \mathtt{c}_i := \mathtt{c}_i - 1$; `goto` p
  - q : `if` $(\mathtt{c}_i = 0)$ `then goto` p `else goto` r
- Configurations $C = (q, n, m)$, with $q \in Q$ and $n, m \in \mathbb{N}$ values of $\mathtt{c}_1$, $\mathtt{c}_2$

Simple Types are Turing-complete!

For $C = (q, n, m)$, let

$$[C] = (\mathtt{S}(\mathtt{q}) \to \mathtt{C}_1(\mathtt{s}_1^n(0_1)) \to \mathtt{C}_2(\mathtt{s}_2^m(0_2)) \to \mathtt{Acc}$$

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$$\begin{array}{lll}
\Gamma_{\mathcal{A}} = \{ & \mathtt{Fin} & : \quad \mathtt{S(q}_F\mathtt{)} \to \mathtt{C}_1(\alpha) \to \mathtt{C}_2(\beta) \to \mathtt{Acc} \\
 & & \\
 & & \bullet\ \mathtt{q} : \mathtt{c}_1 := \mathtt{c}_1 + \mathtt{1};\ \mathtt{goto\ p} \\
 & \mathtt{Add}_1[\mathtt{q},\mathtt{p}] & : \quad (\mathtt{S(p)} \to \mathtt{C}_1(\mathtt{s}_1(\alpha)) \to \mathtt{C}_2(\beta) \to \mathtt{Acc}) \to \\
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 & & \bullet\ \mathtt{q} : \mathtt{if}\ (\mathtt{c}_1 = \mathtt{0})\ \mathtt{then\ goto\ p\ else\ goto\ r} \\
 & \mathtt{Tst}_1^{\mathtt{Z}}[\mathtt{q},\mathtt{p}] & : \quad (\mathtt{S(p)} \to \mathtt{C}_1(\mathtt{0}_1) \to \mathtt{C}_2(\beta) \to \mathtt{Acc}) \to \\
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\end{array}$$

Theorem $\mathcal{A}$ *accepts from* $C \Leftrightarrow \exists e.\ \Gamma_{\mathcal{A}} \vdash e : [C]$

Inhabitation as Logic Programming

- The input theory Γ is the program and search for inhabitants is its execution [4]

[4] Broadly related: [Miller,Nadathur,Pfenning,Scedrov: *Uniform proofs as a foundation for logic programming*, Ann. Pure and Appl. Logic 1991]

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  - *the search for inhabitants is the execution of the program*
  - *the inhabitants are programs generated by synthesis from the interface program as its solution space*

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Finite and bounded CL

- Algorithms, complexity, expressive power of relativized inhabitation with and w/o intersection types
- Connection to computational models: tree automata and alternating Turing machines
  - In distinction to standard CL, we consider the *relativized* inhabitation question, but we bound it
  - In *finite* CL (FCL) we eliminate schematism (monomorphic restriction)
  - In *k-bounded* CL (BCL_k) we bound the depth of types in scheme instantiation (depth k-bounded polymorphism)

Finite and bounded CL

Theorem[Rehof,Urzyczyn TLCA 2011] *For finite combinatory logic we have:*

1. *Relativized inhabitation in* $\textsc{fcl}(\to)$ *is in* $\textsc{Ptime}$
2. *Relativized inhabitation in* $\textsc{fcl}(\to,\cap)$ *is* $\textsc{Exptime}$*-complete*

Theorem[Düdder,Martens,Rehof,Urzyczyn CSL 2012] *For bounded combinatory logic we have:*

1. *Relativized inhabitation in* $\textsc{bcl}_k(\to)$ *is* $\textsc{Exptime}$*-complete for all* k
2. *Relativized inhabitation in* $\textsc{bcl}_k(\to,\cap)$ *is* $(k+2)$-$\textsc{Exptime}$*-complete*

Bounded combinatory logic

- Levels

$$\begin{array}{lcl} \ell(a) & = & 0, \text{ for } a \in \mathbb{A} \cup \mathbb{V}; \\ \ell(\tau \to \sigma) & = & 1 + \max\{\ell(\tau), \ell(\sigma)\}; \\ \ell(\bigcap_{i=1}^{n} \tau_i) & = & \max\{\ell(\tau_i) \mid i = 1, \ldots, n\}. \end{array}$$

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- $\mathrm{BCL}_k(\to, \cap)$

$$\frac{[\ell(S) \leq k]}{\Gamma, x : \tau \vdash_k x : S(\tau)}(\mathsf{var}) \qquad \frac{\Gamma \vdash_k e : \tau \to \tau' \quad \Gamma \vdash_k e' : \tau}{\Gamma \vdash_k (e\ e') : \tau'}(\to\mathsf{E})$$

$$\frac{\Gamma \vdash_k e : \tau_1 \quad \Gamma \vdash_k e : \tau_2}{\Gamma \vdash_k e : \tau_1 \cap \tau_2}(\cap\mathsf{I}) \qquad \frac{\Gamma \vdash_k e : \tau \quad \tau \leq \tau'}{\Gamma \vdash_k e : \tau'}(\leq)$$

Subtyping

$$\sigma \leq \boldsymbol{\omega}, \quad \boldsymbol{\omega} \leq \boldsymbol{\omega} \to \boldsymbol{\omega}, \quad \sigma \cap \tau \leq \sigma, \quad \sigma \cap \tau \leq \tau, \quad \sigma \leq \sigma \cap \sigma;$$
$$(\sigma \to \tau) \cap (\sigma \to \rho) \leq \sigma \to \tau \cap \rho;$$

If $\sigma \leq \sigma'$ and $\tau \leq \tau'$ then $\sigma \cap \tau \leq \sigma' \cap \tau'$ and $\sigma' \to \tau \leq \sigma \to \tau'$

Derived distributivity properties:

$$(\sigma \to \tau) \cap (\sigma \to \rho) = \sigma \to (\tau \cap \rho)$$
$$(\sigma \to \tau) \cap (\sigma' \to \tau') \leq (\sigma \cap \sigma') \to (\tau \cap \tau')$$

Upper bound - ATM algorithm

Input : Γ, τ, k

$$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),\\ x : (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)\}$$
$$\tau = (0 \to 0) \cap (1 \to 1)$$

Upper bound - ATM algorithm

Input : Γ, τ, k

loop :

1 CHOOSE $(x : \sigma) \in \Gamma$;

2 $\sigma' := \bigcap\{S(\sigma) \mid S \in \mathcal{S}_x^{(\Gamma,\tau,k)}\}$;

$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),$
$\quad x : (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)\}$
$\tau = (0 \to 0) \cap (1 \to 1)$

$\sigma' = (0 \to 0) \to (0 \to 0) \to (0 \to 0) \cap \cdots \cap$
$\quad (1 \to 1) \to (1 \to 1) \to (1 \to 1)$

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$\quad (1 \to 1) \to (1 \to 1) \to (1 \to 1)$

3 CHOOSE $m \in \{0, \ldots, \|\sigma'\|\}$;

4 CHOOSE $P \subseteq \mathbb{P}_m(\sigma')$;

$(0 \to 1) \to (1 \to 0) \to (0 \to 0) \cap$
$(1 \to 0) \to (0 \to 1) \to (1 \to 1)$

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5 IF $(\bigcap_{\pi \in P} tgt_m(\pi) \leq \tau)$ THEN

6 IF $(m = 0)$ THEN ACCEPT;

$(0 \to 0) \cap (1 \to 1) \leq \tau$

Upper bound - ATM algorithm

Input : Γ, τ, k

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$(0 \to 0) \cap (1 \to 1) \leq \tau$

6 IF $(m = 0)$ THEN ACCEPT;

7 ELSE

8 FORALL$(i = 1 \ldots m)$

9 $\tau := \bigcap_{\pi \in P} arg_i(\pi)$;

$\tau := (0 \to 1) \cap (1 \to 0)$
$\tau := (1 \to 0) \cap (0 \to 1)$

10 GOTO *loop*;

11 ELSE REJECT;

Upper bound - ATM algorithm

Input : Γ, τ, k

$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),$
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$(x\ f)\ f : (0 \to 0) \cap (1 \to 1)$

Overview

- Theoretical background
- Composition synthesis
- Examples

Situation

- Implemented components in *native language*

```
F ≜
letrec Q : [int] → [int] =
  λx : [int]. if (x = nil) then nil
  else
       (Q (Filter (tail x) (λy: int.(head x) ≤ y))) ::
       (head x) ::
       (Q (Filter (tail x) (λy: int.(head x) > y)))
```

  with *native type*, e.g., $[\texttt{int}] \rightarrow [\texttt{int}]$

- Exposed in semantic repository as combinator $(\texttt{F} : \tau)$ with *semantic type*, e.g.,

$$\tau \quad = \quad ([\texttt{int}] \rightarrow ([\texttt{int}] \cap \mathit{Sorted})) \cap \mathit{SortingFunction.Quicksort}$$

Native vs. Semantic Correctness

- *Native* correctness: for each combinatory interface $(\mathtt{F} : \tau)$ a unique underlying native type $A = \tau^\circ$ and $\Gamma \Vdash e : \tau$, where

$$\Gamma \Vdash e : \tau \quad \textit{if and only if} \quad \Gamma \vdash e : \tau \quad \textit{and} \quad \Gamma^\circ \vdash_s e : \tau^\circ$$

- *Semantic* correctness: *assume* (but do not prove) that semantic types correctly describe semantics of component implementation [5]

[5] [Haack,Howard,Stoughton,Wells: *Fully automatic adaptation of software components based on semantic specifications*, AMAST 2002]

Stratified System

Types

$$\begin{array}{lcll} \mathbb{T}_N & \ni & A, B & ::= \quad \alpha \mid c \mid A \to B \\ & & & \qquad\qquad \alpha, \beta \in \mathbb{V}_N, c \in \mathbb{C}_N \end{array}$$

$$\begin{array}{lcll} \mathbb{T}_S & \ni & \varphi, \psi & ::= \quad \varepsilon \mid d \mid \omega \mid \varphi \to \psi \mid \varphi \cap \psi \\ & & & \qquad\qquad \varepsilon, \eta \in \mathbb{V}_S, d \in \mathbb{C}_S \end{array}$$

$$\begin{array}{lcll} \mathbb{T}_U & \ni & \tau, \sigma & ::= \quad A \mid \tau \to \sigma \mid \tau \cap \varphi \end{array}$$

Underlying native types

$$\begin{array}{lcl} A^\circ & = & A \\ (\tau \to \sigma)^\circ & = & \tau^\circ \to \sigma^\circ \\ (\tau \cap \varphi)^\circ & = & \tau^\circ \end{array}$$

(CL)S

Implementation (B. Düdder & M. Martens, Dortmund):
Combinatory **L**ogic **S**ynthesizer

- Inhabitation algorithms for $\vdash_k$ and $\Vdash_k$
- Integrated into larger framework for specification of semantic repositories and synthesis
- Many optimizations done and more underway
- Experiments
- Deployment on cloud platform (Dortmund) with 2000 cores

Overview

- Theoretical background
- Composition synthesis
- Examples

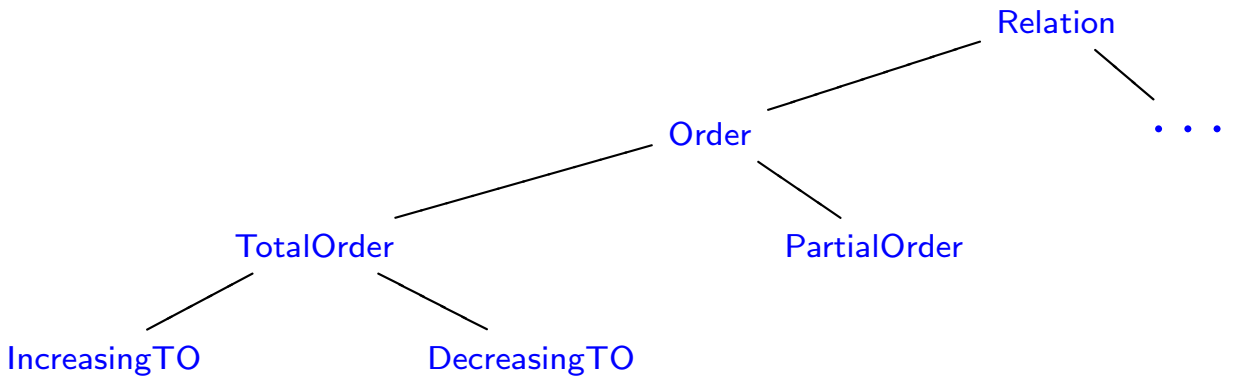
Parameterization

$$
\begin{array}{ll}
\Gamma = \{ & \\
\quad \texttt{F} & : \quad ([\alpha] \to (\alpha \to \texttt{bool}) \to [\alpha]) \cap \mathit{Filter} \\
\quad \texttt{S} & : \quad (([\alpha] \to (\alpha \to \texttt{bool}) \to [\alpha]) \cap \mathit{Filter}) \to \\
 & \qquad ((\alpha \to \alpha \to \texttt{bool}) \to [\alpha] \to [\alpha]) \cap \\
 & \qquad (\mathit{TotalOrder} \to \boldsymbol{\omega} \to \mathit{Sorted}) \cap \\
 & \qquad (\mathit{PartialOrder} \to \boldsymbol{\omega} \to \mathit{TopSorted})) \\
\quad \texttt{G} & : \quad \texttt{Graph}(\alpha) \to ((\alpha \to \alpha \to \texttt{bool}) \cap \mathit{PartialOrder}) \\
\quad \texttt{N} & : \quad \texttt{Graph}(\alpha) \to [\alpha] \\
\quad \circ & : \quad (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma) \\
\quad \diamond & : \quad (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma) \\
\} &
\end{array}
$$

$\diamond = \mathbf{S} = \lambda x.\lambda y.\lambda z.xz(yz)$

$\circ = \mathbf{B}' = \lambda x.\lambda y.\lambda z.y(xz)$

Semantic Structures



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\}
\end{array}
$$

Inhabitation goal

$$\Gamma \Vdash ? : \texttt{Graph}(\alpha) \to ([\alpha] \cap \mathit{TopSorted})$$

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\quad \diamond \quad : \quad (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma) \\
\}
\end{array}
$$

Inhabitation goal

$$\Gamma \Vdash ? : \texttt{Graph}(\alpha) \to ([\alpha] \cap \mathit{TopSorted})$$

with solution

$$\Gamma \Vdash (\texttt{G} \circ (\texttt{S}\ \texttt{F})) \diamond \texttt{N} : \texttt{Graph}(\alpha) \to ([\alpha] \cap \mathit{TopSorted})$$

Service Composition

$$
\begin{array}{ll}
\Gamma = \{ & \\
\quad \texttt{F} & : \quad ([\alpha] \to (\alpha \to \texttt{bool}) \to [\alpha]) \cap \mathit{Filter} \\
\quad \texttt{S} & : \quad (([\alpha] \to (\alpha \to \texttt{bool}) \to [\alpha]) \cap \mathit{Filter}) \to \\
 & \qquad ((\alpha \to \alpha \to \texttt{bool}) \to [\alpha] \to [\alpha]) \cap \\
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\quad \circ & : \quad (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma) \\
\quad \diamond & : \quad (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma) \\
\quad \texttt{MyID} & : \quad () \to (\texttt{int} \cap \mathit{UserID}) \\
\quad \texttt{GetTasks} & : \quad () \to \texttt{Graph}(\texttt{Task}) \\
\} &
\end{array}
$$

$$
\begin{array}{ll}
\Delta = \{ & \\
\quad \texttt{S} :: \texttt{Connect} & : \quad () \to (\texttt{int} \cap \mathit{SessionID}) \\
\quad \texttt{S} :: \texttt{ReqTransaction} & : \quad (\texttt{int} \cap \mathit{UserID}) \to (\texttt{int} \cap \mathit{SessionID}) \to \\
 & \qquad ([\texttt{Task}] \cap \mathit{TopSorted}) \to (\texttt{int} \cap \mathit{TID}) \\
\quad \texttt{S} :: \texttt{EndTransaction} & : \quad (\texttt{int} \cap \mathit{TID}) \to [\texttt{int} \cap \mathit{Result}] \\
\} &
\end{array}
$$

Service Composition

We can now ask for the computation of a result list by the inhabitation goal

$$\Gamma \cup \Delta \vdash ? : [\texttt{int} \cap \mathit{Result}]$$

Service Composition

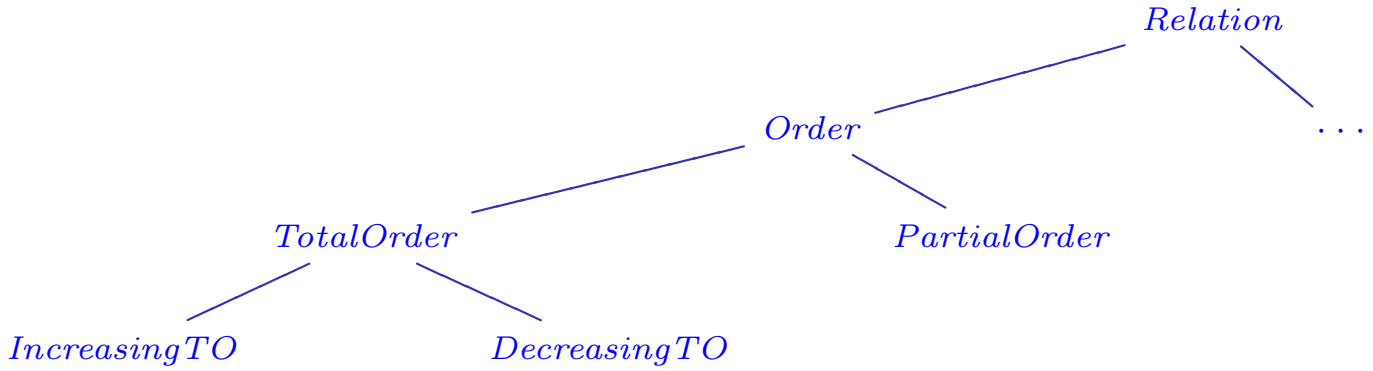
We can now ask for the computation of a result list by the inhabitation goal

$$\Gamma \cup \Delta \Vdash ? : [\texttt{int} \cap \mathit{Result}]$$

which generates the solution

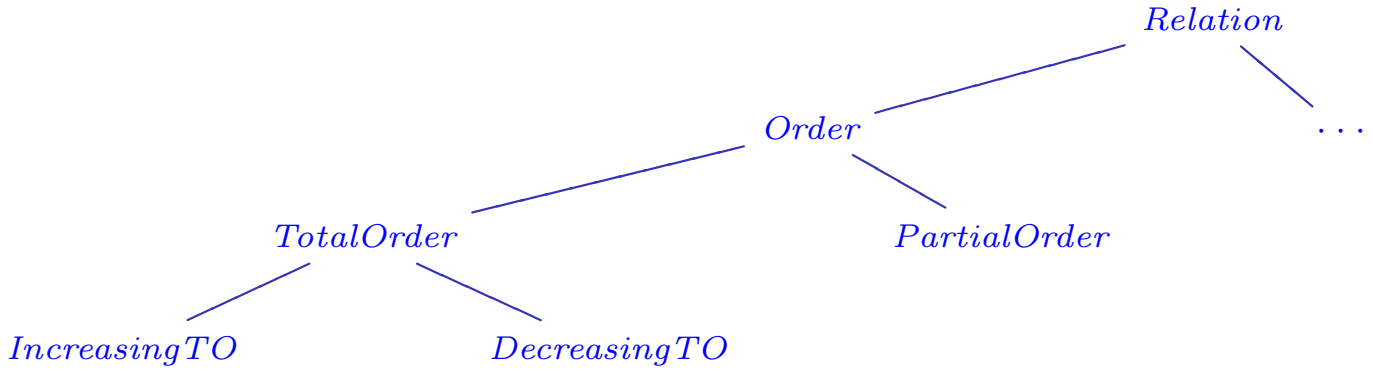
$$\begin{array}{l}\Gamma \cup \Delta \Vdash \texttt{let X} = \texttt{S} :: \texttt{ReqTransaction} \\ \qquad\qquad\qquad\qquad\qquad\qquad (\texttt{MyID}()) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad (\texttt{S} :: \texttt{Connect}()) \\ \qquad\qquad \texttt{let Y} = (\texttt{G} \circ (\texttt{S F})) \diamond \texttt{N} \\ \qquad\qquad\quad \texttt{in} \quad (\texttt{X} \circ \texttt{Y} \circ \texttt{EndTransaction})\ (\texttt{GetTasks}()) \\ \qquad\qquad : [\texttt{int} \cap \mathit{Result}]\end{array}$$

Logic



- $Rev(R) = \{r^{-1} \mid r \in R\}, r^{-1} = \{(y, x) \mid (x, y) \in r\}$

Logic



- $Rev(R) = \{r^{-1} \mid r \in R\}, r^{-1} = \{(y, x) \mid (x, y) \in r\}$
  - $Rev(Relation.DecreasingTO) = Relation.IncreasingTO$
  - $Rev(Relation.IncreasingTO) = Relation.DecreasingTO$

Logic

$$
\begin{array}{lcl}
\Gamma = \{ & & \\
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 & & \quad (\mathit{TotalOrder}.\varepsilon \to \boldsymbol{\omega} \to \mathit{Sorted}(\varepsilon)) \cap \\
 & & \quad (\mathit{PartialOrder}.\eta \to \boldsymbol{\omega} \to \mathit{TopSorted}(\eta))) \\
\quad \ldots & & \\
\quad \texttt{swap} & : & ((\alpha \to \beta \to \gamma) \to (\beta \to \alpha \to \gamma)) \cap \\
 & & (\mathit{Relation}.\varepsilon \to \mathit{Rev}(\mathit{Relation}.\varepsilon)) \\
\quad \leq_{\mathbb{N}} & : & (\texttt{int} \to \texttt{int} \to \texttt{bool}) \cap \mathit{Relation.IncreasingTO} \\
\quad \Phi_{\texttt{Rev}} & : & (\alpha \to \alpha) \cap \\
 & & (\mathit{Rev}(\mathit{Relation.IncreasingTO}) \to \mathit{Relation.DecreasingTO}) \cap \\
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$$

Inhabitation goal

$$\Gamma \Vdash ? : [\texttt{int}] \to ([\texttt{int}] \cap \mathit{Sorted}(\mathit{DecreasingTO}))$$

Logic

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 & & (\mathit{Rev}(\mathit{Relation.IncreasingTO}) \to \mathit{Relation.DecreasingTO}) \cap \\
 & & (\mathit{Rev}(\mathit{Relation.DecreasingTO}) \to \mathit{Relation.IncreasingTO}) \\
\} & &
\end{array}
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Inhabitation goal

$$\Gamma \Vdash ? : [\texttt{int}] \to ([\texttt{int}] \cap \mathit{Sorted}(\mathit{DecreasingTO}))$$

Solution

$$
\begin{array}{l}
\Gamma \Vdash \texttt{S F} \; (\Phi_{\texttt{Rev}} \; (\texttt{swap} \leq_{\mathbb{N}})) : \\
\qquad [\texttt{int}] \to ([\texttt{int}] \cap \mathit{Sorted}(\mathit{DecreasingTO}))
\end{array}
$$

End: Ongoing and Future Work

- Applications
  - Optimizations of **CLS**-algorithm
  - Experiments
  - Staged system
- Further theoretical issues
  - Algorithmic properties of $\leq$
  - Connections to temporal logic synthesis (2-EXPTIME & BCL_0)
  - Process types?